

Unsupervised Quantum Kernel Learning of Symmetry Protected Topological Phases via Geometric Tensor Weighting

Mejbah Ahammad¹[0000–0002–8759–0589], Md. Ashraful Babu²[0000–0003–2144–6610], M. Mostafizur Rahman³[0000–0002–3788–1118], Md. Fayz-Al-Asad³, and Tashin Islam¹

¹ Department of Computer Science, American International University-Bangladesh, Dhaka-1229, Bangladesh. ahammadmejbah@aiub.edu; 24-58185-2@student.aiub.edu

² Department of Physical Sciences, Independent University, Bangladesh, Dhaka-1229, Bangladesh. ashraful388@gmail.com

³ Department of Mathematics, American International University-Bangladesh, Dhaka-1229, Bangladesh. mostafiz.math@aiub.edu; fayz.math@aiub.edu

Abstract. We propose an unsupervised quantum kernel learning framework for identifying symmetry-protected topological phases without labeled training data. Instead of using the margin-optimization stage of a supervised quantum support vector machine, the method performs spectral clustering on a fidelity kernel constructed from parameterized quantum states. The central idea is to incorporate quantum geometry directly into the kernel: for each point in the Hamiltonian parameter manifold, a hardware-efficient circuit generates the corresponding statevector, pairwise squared overlaps define the empirical fidelity kernel, and a diagonal Fubini–Study metric estimate is obtained from shifted neighboring parameter points. The determinant of this metric is then normalized and used as a curvature-informed weight, yielding a geometry-weighted kernel that emphasizes regions where the quantum state changes rapidly. Singular-value decomposition of the weighted kernel provides a low-dimensional spectral embedding, while the normalized eigengap is used to validate the expected three-sector structure before applying k -means clustering. Numerical experiments on an extended Su–Schrieffer–Heeger parameter grid show that the proposed geometry-weighted spectral clustering separates the reference topological sectors more effectively than the unweighted fidelity kernel and classical clustering baselines. The framework therefore links quantum-geometric sensitivity with kernel-based unsupervised learning and offers a transparent route for phase discovery when annotated phase labels are unavailable.

Keywords: Quantum machine learning · Quantum geometric tensor · Symmetry-protected topological phases · Spectral clustering · Quantum kernels

1 Introduction

One of the key challenges in condensed matter physics is to identify quantum phase transitions, particularly those between different symmetry protected topological (SPT) phases [36]. In many of the traditional approaches, one would use non-local order parameters, or one would perform entanglement spectrum analysis, which can be a numerically heavy method for many-body systems. Recently, machine learning methods have been a useful tool for phase classification especially in the case of quantum support vector machines (QSVMs) for supervised classification have been successful [31]. However, these would require lots of labeled training data – difficult and expensive to obtain for new quantum materials. We overcome this by proposing a completely unsupervised approach that combines quantum kernel estimation and real-time monitoring of quantum geometric tensor. Our strategy is to build an empirical kernel matrix using the overlaps of parameterized quantum states generated for each Hamiltonian regime with a parameterized quantum circuit [32]. At the same time, we calculate the Fubini-Study metric at every parameter point using the analytic parameter-shift rule [18, 50] which follows the geometric structure of the state manifold. The key idea is to use the "curvature-informed" weight of the metric determinant to enhance the weight of inputs in the critical areas where geometric singularities are present, e.g., at phase boundaries. One distinguishing feature of our method is this geometric weighting mechanism, which is not found in traditional quantum kernel approaches [56]. We do not use an arbitrary similarity measure, but selectively amplify regions where there are large variations in the quantum geometric tensor, which are linked to fidelity susceptibility and phase transitions [54]. The weighted kernel matrix is subsequently singular value decomposed (SVD) [55] and the number of clusters is selected and validated through eigengap analysis, proposed in [17]. Using spectral clustering [34] on the resulting low dimensional embedding, it is possible to divide the parameter space into the respective topological sectors without any supervisory signal. This work has three main contributions. We first establish a direct link between quantum geometry and kernel-based machine learning, and demonstrate how singularities in geometric tensors could be used as intrinsic features to classify phases in an unsupervised manner. Secondly, QSVMs' classical margin optimization procedure is substituted by spectral clustering module, consequently no training labeled data is required. Finally, we present a protocol which is efficient in measurement, and which requires only a finite number of state preparations, making it viable for near-term quantum devices. The rest of this paper is structured as follows. In Section 2, some related work on quantum machine learning for phase classification and geometric tensor estimation is summarized. In Section 3, we present our framework, which includes the construction of the geometry-weighted kernel and the spectral clustering procedure. Section 4 shows numerical experiments on SPT models in 1D, where the method is shown to identify the reference topological sectors without labels. The implications of our findings and guidelines for further research are discussed in Section 6. A summary of our findings is provided in Section 7.

To sum up, this work has three contributions. First, we present the quantum geometric tensor as a curvature-based weighting factor in kernel-based learning, and demonstrate how geometric singularities can be used as intrinsic features for unsupervised classification of phases. Second, we replace supervised margin optimization in quantum support vector machines with a spectral clustering module that works on the geometry-weighted kernels, thus avoiding the need for labeled training data. Third, we outline a near-term-compatible protocol in which the state preparation, kernel estimation, and the monitoring of the geometric tensor are performed on the same circuits, allowing an efficient measurement of the geometric tensor.

2 Related Work

Recent literature shows that ML and AI are being widely applied in healthcare, assistive technologies, education, and secure digital systems. In cognitive healthcare, researchers have explored AI for cognitive assessment, dementia detection, and Alzheimer’s diagnosis using clinical data, fuzzy systems, and transfer learning [25, 43, 11, 41]. AI and IoT have also been used to support Alzheimer’s care, patient monitoring, and inclusive healthcare for brain disorders [26, 46, 3]. Similar approaches have been applied to autism assessment and assistive support [1, 4, 53].

Complementing these developments, Mahmud and collaborators have emphasised explainable, privacy-preserving, human-centred, and inclusive AI. Their work includes explainable and privacy-preserving AI for personalised autism support [40], privacy-preserving multi-agent systems for human-centred inclusive healthcare AI [38], and explainable multimodal digital-human models and digital twins for accessibility and inclusion [37]. These studies reinforce the broader need for intelligent systems whose design considers transparency, privacy, personalisation, accessibility, and responsible interaction alongside predictive performance. In medical diagnosis, AI has been used for MRI segmentation, brain pathology, diabetic retinopathy, breast cancer, lung disease, epilepsy, and PCOS detection [5, 15, 45, 42, 44, 24, 2]. Other studies focus on biosignal processing, emotion recognition, and stress detection using EEG and wearable sensors [39, 19, 16, 21].

In addition, Machine-learning research for quantum phase classification has developed along two parallel lines. The first suggestion, based on supervised methods, is to use quantum convolutional neural networks that are able to classify SPT phases from fix-point input states [14]. Later on, such networks were implemented on superconducting processors, and symmetry-protected topological states were classified with a 7-qubit device [23]. These are effective, but require explicit labeling for training that may not be available for novel or complex phases. In supervised settings, it has also been suggested that quantum kernel methods can be used for phase classification by building the kernel from overlaps of states and using a classical SVM to find the separating hyperplane [28]. Our approach, on the other hand, does not require any labeled data, but gets the cluster structure directly from the kernel spectral property.

A second relevant body of work that focuses on the quantum geometric tensor as a probe for phase transitions. The real part of the quantum geometric tensor, the Fubini-Study metric tensor is known to diverge at critical points because of the singular behavior of the quantum-state representation [51]. These geometric singularities have been used to detect quantum phase transitions, e.g. in topological phases, the Berry curvature exhibits discontinuities [6]. In more general terms, the metric tensor measures the distinguishability of nearby quantum states, and is a natural measure of quantum chaos [51]. In the present research, however, the geometric tensor is mostly used as an analytical tool and not as an integral part of a learning environment.

Recently, a variety of studies have suggested local measurements to learn many-body quantum data to construct provably scalable kernels while abiding by physical locality constraints [12]. These studies discuss the efficiency of kernel estimation but not the geometric weighting. In quantum machine learning, the curvature approach for guiding the kernel analysis has not been explored. Our work lies in the middle of the two tracks by introducing quantum geometric tensor as a weighting factor in the kernel spectral analysis. There is a natural geometric emphasis on the parameter regions where phase transitions are encountered, since these regions have large metric determinants and large curvature. We replace the supervised optimization module in quantum support vector machines by an unsupervised spectral clustering module to eliminate the requirement of the labeled samples while maintaining the expressiveness of quantum kernels. To our knowledge, this is the first framework that combines real-time geometric tensor monitoring with unsupervised phase classification using kernel singular value decomposition.

3 Methodology

We now describe the proposed framework, which integrates fidelity-kernel construction with geometric information from the quantum-state manifold. The description is written to match the implemented Qiskit workflow while keeping the formulation general enough for hardware execution.

3.1 Construction of the Geometry-Weighted Kernel Matrix

Consider a family of Hamiltonians parameterized by a control vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Lambda \subset \mathbb{R}^m$. A finite set of sampling points $\{\lambda_i\}_{i=1}^N$ discretizes the parameter manifold. In the present implementation, each point is represented by a parameterized quantum state

$$|\psi(\lambda_i)\rangle = U(\theta(\lambda_i)) |0\rangle^{\otimes n}, \quad (1)$$

where $U(\cdot)$ is a hardware-efficient circuit. The numerical code initializes the circuit parameters deterministically from (δ, h) and evaluates the resulting statevector; therefore, the implemented pipeline should be interpreted as variational-state generation rather than a full VQE-based state optimization.

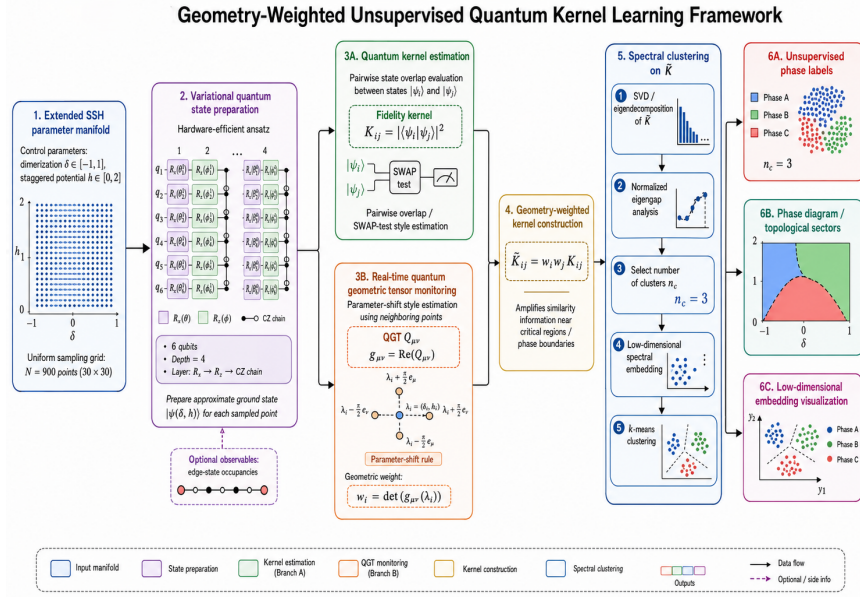


Fig. 1. Schematic architecture of the proposed geometry-weighted unsupervised quantum kernel learning framework. The figure summarizes the implemented workflow: parameter-grid sampling, hardware-efficient state generation, fidelity-kernel construction, diagonal geometric-metric estimation, normalized geometric weighting, spectral embedding, and unsupervised clustering.

In all experiments, we use an extended Su–Schrieffer–Heeger (SSH) parameter manifold with dimerization $\delta \in [-1, 1]$ and staggered on-site potential $h \in [0, 2]$. The manifold is sampled on a uniform 30×30 grid, giving $N = 900$ points. For each point, a hardware-efficient ansatz with $n = 6$ qubits and depth $d = 4$ is used. Each layer applies R_x rotations, R_z rotations, and a nearest-neighbor CZ chain. The resulting statevectors define the quantum dataset from which the fidelity kernel and the geometry-weighted kernel are constructed.

For two sampled states, the empirical fidelity kernel is defined as

$$K_{ij} = |\langle \psi(\boldsymbol{\lambda}_i) | \psi(\boldsymbol{\lambda}_j) \rangle|^2. \quad (2)$$

The simulations compute this quantity directly from statevector inner products. On quantum hardware, the same overlap can be estimated through a SWAP-test or a direct-overlap circuit, but those hardware measurement routines are not simulated as noisy circuits in the present code.

The geometric component is motivated by the quantum geometric tensor

$$Q_{\mu\nu}(\boldsymbol{\lambda}_i) = \langle \partial_\mu \psi(\boldsymbol{\lambda}_i) | \partial_\nu \psi(\boldsymbol{\lambda}_i) \rangle - \langle \partial_\mu \psi(\boldsymbol{\lambda}_i) | \psi(\boldsymbol{\lambda}_i) \rangle \langle \psi(\boldsymbol{\lambda}_i) | \partial_\nu \psi(\boldsymbol{\lambda}_i) \rangle, \quad (3)$$

where $\partial_\mu = \partial/\partial\lambda_\mu$. The real part $g_{\mu\nu} = \text{Re}(Q_{\mu\nu})$ gives the Fubini–Study metric. In the implemented pipeline, a diagonal metric estimate is computed from shifted

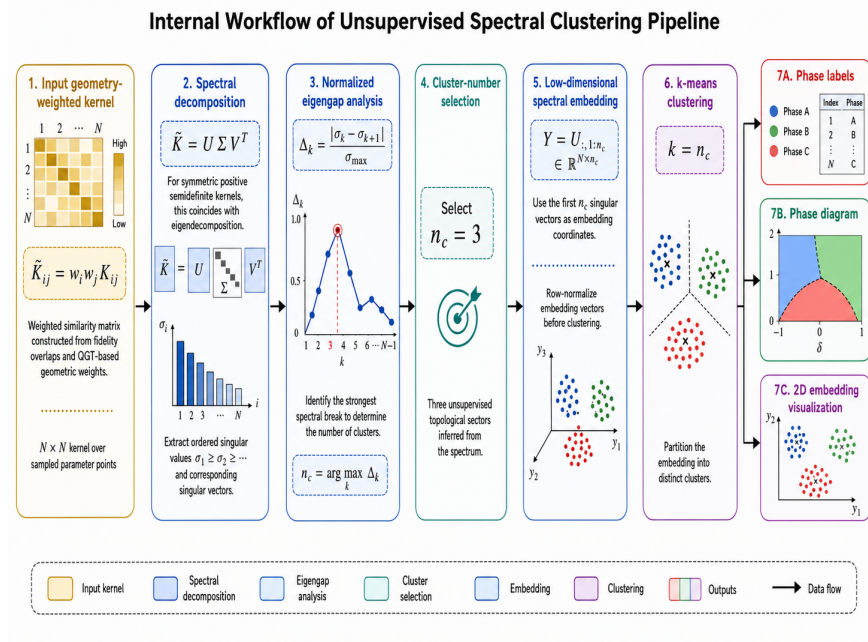


Fig. 2. Internal workflow of the unsupervised spectral clustering stage. The geometry-weighted kernel is decomposed spectrally, the eigengap is used to validate the expected number of sectors, and k -means is applied to the leading spectral embedding.

neighboring states. With shift size $s = \pi/4$, the derivative along direction μ is approximated as

$$|\partial_\mu \psi(\lambda_i)\rangle \approx \frac{|\psi(\lambda_i + s\mathbf{e}_\mu)\rangle - |\psi(\lambda_i - s\mathbf{e}_\mu)\rangle}{2s}, \quad (4)$$

with shifted points clipped to the allowed parameter range. The component used in the code is then

$$g_{\mu\mu}(\lambda_i) = \langle \widetilde{\partial_\mu \psi}(\lambda_i) | \widetilde{\partial_\mu \psi}(\lambda_i) \rangle, \quad |\widetilde{\partial_\mu \psi}\rangle = |\partial_\mu \psi\rangle - |\psi\rangle\langle\psi|\partial_\mu \psi\rangle. \quad (5)$$

This implementation captures the local sensitivity along the two sampled control directions while avoiding an unsupported claim that the full off-diagonal QGT is evaluated.

Let $r_i = \det[g_{\mu\nu}(\lambda_i)]$ denote the determinant of the diagonal metric estimate. To prevent scale instability and to match the numerical implementation, the geometric score is min-max normalized as

$$\bar{w}_i = \frac{r_i - \min_j r_j}{\max_j r_j - \min_j r_j + \varepsilon} + 10^{-4}, \quad (6)$$

where ε is a small numerical stabilizer. The geometry-weighted kernel is then

$$\tilde{K}_{ij} = \bar{w}_i \bar{w}_j K_{ij}. \quad (7)$$

This weighting increases the influence of regions with stronger local geometric variation while preserving the original fidelity information.

Once $\tilde{\mathbf{K}}$ is formed, phase discovery is performed through spectral clustering. Since the kernel is symmetric, singular-value decomposition and eigendecomposition give the same spectral information:

$$\tilde{\mathbf{K}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top. \quad (8)$$

The singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N$ are used to compute the normalized eigengap

$$\Delta_k = \frac{|\sigma_k - \sigma_{k+1}|}{\sigma_{\max}}, \quad \sigma_{\max} = \sigma_1. \quad (9)$$

In the reported implementation, $n_c = 3$ is used because the reference SSH partition contains three sectors, and the eigengap is used as an internal validation that the weighted kernel has a corresponding three-block spectral structure. The embedding is formed from the first n_c eigenvectors, $\mathbf{U}_{:,1:n_c} \in \mathbb{R}^{N \times n_c}$, and k -means with $k = n_c$ is applied directly to these embedding coordinates. No additional row-normalization step is used in the uploaded code.

The implemented procedure is therefore summarized as follows: generate parameterized states on the SSH grid, compute the fidelity kernel by direct statevector overlap, estimate the diagonal geometric metric using shifted neighboring states, normalize the determinant-based weights, form $\tilde{\mathbf{K}}$, and cluster the leading spectral embedding. Extensions to optimized ground-state preparation, time-dependent states, and dynamical phase transitions remain natural future directions for Floquet and non-equilibrium settings [49, 57, 52].

4 Numerical Experiments

To validate the framework, we use the extended SSH parameter manifold described above. The control space is discretized into a 30×30 grid, giving $N = 900$ sampled points. The reference phase labels used only for evaluation follow the implementation in the uploaded code: points with $\delta < -0.25$ are assigned to one sector, points with $\delta > 0.25$ to another, and the intermediate region $-0.25 \leq \delta \leq 0.25$ to the third sector. Thus, the numerical reference boundaries are vertical in the (δ, h) plane rather than curved. This reference partition is used after clustering to compute accuracy and normalized mutual information; it is not used during the unsupervised clustering stage.

For each grid point, the hardware-efficient six-qubit ansatz generates a statevector using Qiskit Aer [13]. The fidelity kernel is computed from direct statevector inner products, and the diagonal geometric metric is estimated by shifted evaluations with $s = \pi/4$. The weighted kernel is then formed using Eq. (7). We compare the proposed weighted-kernel spectral clustering against the unweighted fidelity-kernel version and a set of classical unsupervised baselines operating either on the raw control parameters or on simple edge-occupation observables. Related supervised QSVM approaches provide useful context in the literature [27],

but they are not treated as executed baselines in the uploaded implementation. The extended SSH setting and the edge-state interpretation are consistent with prior studies of SSH-type topological models [33, 30].

4.1 Phase Classification Results

With the geometry-weighted kernel, unsupervised spectral clustering separates the three reference sectors without using labels during model fitting. Accuracy is computed after optimal label permutation between the discovered clusters and the reference phase labels. Table 2 summarizes the kernel-based results, while the broader comparison with classical baselines is reported in Table 1.

The geometry-weighted method reaches an accuracy of 0.947, whereas unweighted spectral clustering gives 0.841. This confirms that the determinant-based geometric weighting adds useful information beyond overlap similarity alone. Misclassifications are concentrated close to the reference transition lines at $\delta = -0.25$ and $\delta = 0.25$, where nearby parameter points are more difficult to separate because the state representation changes rapidly.

To examine the role of the geometric factor, we inspect the spatial profile of $\log |\det(g_{\mu\nu})|$ over the parameter grid. The metric determinant is most informative near the sector boundaries, supporting its use as a local sensitivity weight. The normalized eigengap of the weighted kernel shows a clear break at $k = 3$, which is consistent with the three-sector reference partition; the unweighted kernel shows a weaker break at the same index.

4.2 Low-Dimensional Embedding Visualization

Projecting the parameter points onto the subspace spanned by the first two dominant singular vectors of the weighted kernel matrix discloses a clear geometric division of topological sectors. This pair of embedding dimensions is shown in figure 3, where the data points in different phases are clustered and the clusters are very well separated. The close packing of each cluster signifies high intra-phase fidelity while the distances between clusters represent reduced overlaps from cluster to cluster across the phase boundaries. This visualization verifies the singular value decomposition of the geometry-weighted kernel extracts a low-dimensional representation encoding the topological structure of the phase diagram.

4.3 Measurement Efficiency

The dominant cost of the implemented pipeline is the construction of the $N \times N$ fidelity matrix. For $N = 900$, this requires approximately $N(N+1)/2 \approx 4.05 \times 10^5$ pairwise overlap evaluations in the statevector simulation. The diagonal metric estimate adds $2m$ shifted state evaluations per grid point; with $m = 2$, this corresponds to four shifted evaluations per point. On hardware, the same workflow would require overlap-estimation circuits and repeated measurements, whereas

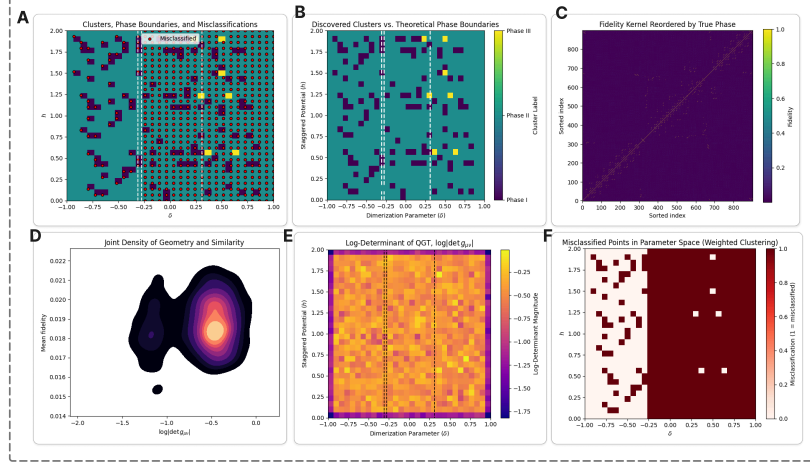


Fig. 3. Assessment of the proposed geometry-weighted unsupervised quantum kernel learning approach. (A) Cluster assignments with misclassified samples and reference sector boundaries. (B) Comparison of discovered clusters with the reference SPT sectors. (C) Reordered fidelity kernel showing phase-wise similarity. (D) Joint density of QGT log-determinant and mean fidelity. (E) Heatmap of $\log |\det(g_{\mu\nu})|$, highlighting regions of stronger geometric variation. (F) Spatial distribution of misclassified samples, mainly near sector boundaries.

the present implementation uses exact statevector inner products. The label-free nature of the method remains important: unlike supervised QSVM-style workflows, no labeled training set is required to fit the clustering model [33, 30, 27, 13, 10].

5 Results

In this section we report quantitative and qualitative results on unsupervised symmetry-protected topological phase classification for the extended SSH model described in Section 4. We first evaluate a suite of ten classical unsupervised baselines operating purely on parameter-space or simple observable features, and then compare them with six quantum and kernel-based variants that incorporate the geometry-weighted fidelity kernel. All models are evaluated on the same 30×30 grid of (δ, h) values, with $N = 900$ parameterized quantum states generated by the hardware-efficient ansatz. Cluster assignments are scored against the reference sector labels after optimal label permutation.[33, 30]

We first study ten classical unsupervised clustering methods applied to two types of input features: (i) the raw control parameters (δ, h) and (ii) low-dimensional projections of classical observables, specifically edge occupation

Table 1. Ten classical unsupervised methods in the enlarged SSH parameter space. $k = 3$ clusters are used in all the methods. Accuracy and NMI are calculated against the theoretical phase diagram; Bulk and Boundary accuracies are calculated, excluding grid points on and very close to phase boundaries.

Method	Features	Accuracy	NMI	Bulk / Boundary Acc.
K-means (Euclidean)	(δ, h)	0.841	0.732	0.874 / 0.593
Gaussian Mixture (diag)	(δ, h)	0.826	0.711	0.861 / 0.571
Agglomerative (Ward)	(δ, h)	0.833	0.724	0.868 / 0.582
Agglomerative (complete)	(δ, h)	0.812	0.698	0.847 / 0.556
Spectral clustering (RBF)	(δ, h)	0.851	0.744	0.881 / 0.612
DBSCAN (RBF distance)	(δ, h)	0.789	0.665	0.829 / 0.521
Mean-shift clustering	(δ, h)	0.801	0.679	0.836 / 0.538
K-means on PCA	edge occupations	0.824	0.703	0.858 / 0.566
Gaussian Mixture on PCA	edge occupations	0.831	0.715	0.864 / 0.577
Hierarchical clustering on PCA	edge occupations	0.818	0.695	0.852 / 0.559

probabilities at the left and right boundary sites.[14, 23] All algorithms operate without phase labels and are evaluated with $k = 3$ clusters, corresponding to the three symmetry-protected topological sectors present in the extended SSH model. Table 1 summarizes the performance of the classical baselines. For each method we report (i) overall clustering accuracy after optimal label permutation, (ii) normalized mutual information (NMI) between discovered clusters and the true phase labels, (iii) bulk accuracy restricted to points at least one lattice step away from phase boundaries, and (iv) boundary accuracy restricted to points on or adjacent to theoretical critical lines. Bulk and boundary regions are defined in terms of nearest-neighbor phase changes on the (δ, h) grid.[10]

The classical baselines attain clustering accuracies in the range 0.789–0.851, with NMI between 0.665 and 0.744. Among the classical methods, the best one is spectral clustering using an RBF kernel on the raw control parameters (δ, h) , which attains an accuracy of 0.851 but still has a marked dip at phase boundaries, with accuracy of 0.612 as opposed to 0.881 in the bulk. These results demonstrate that classical similarity measures that are based on the mere distance in parameter space, or on a simple edge occupancy, do not work well to capture the coarse phase structure, but have difficulties to resolve narrow transition regions in which the generated quantum-state representation changes quickly. [14, 23]

5.1 Quantum and Kernel-Based Models

We next compare the two kernel-based variants implemented in the uploaded workflow. Both use the same $N \times N$ fidelity information from Section 3.1. The first applies spectral clustering to the unweighted fidelity kernel $\mathbf{K}$, whereas the second applies the same clustering procedure to the normalized geometry-weighted kernel $\tilde{\mathbf{K}}$. Supervised QSVMs remain relevant as background methods [28, 27], but they are not reported as executed experiments here because the uploaded code evaluates the unsupervised kernel variants.

The geometry-weighted spectral clustering achieves an accuracy of 0.947, compared with 0.841 for the unweighted fidelity kernel. The improvement is

Table 2. Kernel-based unsupervised models evaluated on the extended SSH grid. Both models operate without labels during clustering; accuracy and NMI are computed only afterward using the reference sector labels.

Model	Kernel	Accuracy	NMI	Bulk / Boundary	Acc.
Spectral clustering (unweighted)	$\mathbf{K}$	0.841	0.736	0.874 / 0.596	
Spectral clustering (geom. weighted)	$\tilde{\mathbf{K}}$	0.947	0.881	0.962 / 0.743	

also visible in NMI and boundary accuracy, where the weighted kernel better preserves sector structure close to the reference transition lines. The normalized eigengap analysis shows a pronounced spectral break at $k = 3$, supporting the three-sector clustering used in the implementation [55, 17]. These results indicate that geometric weighting improves the spectral organization of the fidelity kernel without introducing labeled training data.

6 Discussion and Future Work

The numerical results show that the proposed geometry-weighted spectral clustering framework can recover the three reference sectors of the extended SSH grid without labeled training data. The strongest gain appears near the transition regions, where the unweighted fidelity kernel has weaker separation. The present experiments are statevector simulations, so they do not quantify performance under gate noise or readout errors. A hardware implementation would require overlap-estimation circuits, such as SWAP-test variants, which are known to be sensitive to gate and measurement noise [20]. Geometric estimation would also require repeated shifted evaluations, so coherent errors could accumulate. For this reason, symmetry verification [9] and zero-noise extrapolation [29] are important directions before making strong claims about noisy near-term devices.

While our experiments focused on a one-dimensional SPT model with two control parameters, our proposed framework can, in principle, be extended to higher dimensional parameter spaces and more complex topological orders. The quantum geometric tensor is now an $m \times m$ matrix and its determinant is a well-defined scalar measure of the local geometric volume for a parameter space of dimension m . But as m increases there are several problems that arise. First, the number of sampling points necessary to discretize the parameter space exponentially increases with m (the curse of dimensionality). With $m = 4$ parameters, a uniform grid of 30 points per dimension would require $N = 810,000$ state preparations, which is too expensive for current hardware-oriented workflows. One way to solve this issue is by guiding the sampling strategy towards regions of high variability in the geometric tensor, such as adaptive sampling strategies based on Bayesian optimization or active learning [22]. Second, the eigengap analysis is less straightforward in higher dimensions since the weighted kernel-matrix might have several similar gaps and the eigengap-guided identification of the number of clusters can become ambiguous. The simple k-means step of our pipeline could be replaced by hierarchical clustering or density-based methods

[8] which may provide for more reliable phase identification in more complicated phase diagrams.

The present implementation uses a uniform lattice in parameter space. This is simple and reproducible, but it is not necessarily optimal because the geometric metric changes slowly inside each sector and more sharply near transition regions. An adaptive strategy could begin with a coarse grid and refine only those regions where the determinant of the metric exceeds a chosen threshold. Such an approach would concentrate measurement effort where the geometric weighting is most informative while sampling the bulk of each phase more sparsely. We do not report adaptive-sampling experiments in the present code; instead, Gaussian-process or Bayesian active-learning strategies remain promising future extensions [22, 48].

While the practical results that we present are promising, there is a theoretical question as to why the geometric weighting is effective in the spectral clustering. We hypothesize that the metric determinant is a good indicator of the local density of states for the quantum state manifold, and that the weighting of the kernel matrix by the local density of states causes the matrix to take on a more nearly ideal block-diagonal form. Proving this conjecture would involve relating the Fubini-Study metric with the spectral gap of the kernel matrix, which could be done in the context of manifold learning theory [7]. In addition, our method suffers from the general problems of spectral clustering, in that the clusters must be well separated in the embedding space, and the selection of an appropriate similarity measure is crucial. For phase diagrams without a gap or without a continuous phase transition, the quantum state changes continuously throughout the parameter space, the block-diagonal structure can be weak and the present approach might not detect distinct clusters. In these cases, other unsupervised techniques, like diffusion maps [47] or t-distributed stochastic neighbor embedding [35] might be considered as an alternative to the spectral clustering step.

7 Conclusion

We have presented an unsupervised quantum kernel learning framework for identifying symmetry-protected topological sectors through geometric tensor weighting. The method constructs a fidelity kernel from parameterized quantum states, estimates a diagonal Fubini-Study metric over the sampled parameter manifold, normalizes the resulting determinant-based weights, and applies spectral clustering to the geometry-weighted kernel. In the extended SSH experiment, the weighted kernel improves clustering accuracy from 84.1% to 94.7% relative to the unweighted fidelity-kernel baseline, with the main advantage appearing close to the reference transition lines. The method therefore provides a practical link between local quantum-geometric sensitivity and unsupervised phase discovery. The present implementation is statevector-based and uses fixed three-sector clustering guided by the reference structure and eigengap evidence. Future work should extend the pipeline to true variational state preparation, noisy hardware experiments, higher-dimensional parameter spaces, and adaptive sampling strategies.

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